W44. Compute

$$\lim_{n\to\infty}\left\{n\left[\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\sqrt{2}}\right)\left(1+\frac{1}{n\sqrt{3}}\right)\cdots\left(1+\frac{1}{n\sqrt{n}}\right)-1\right]-2\sqrt{n}\right\}$$

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Let $P_n := \prod_{k=1}^{n}\left(1+\frac{1}{n\sqrt{k}}\right), S_n := \sum_{k=1}^{n}\frac{1}{\sqrt{k}}, H_n := \sum_{k=1}^{n}\frac{1}{k}$ and let

$\sigma_k = \sigma_k(x_1,x_2,\ldots,x_n) := \sum_{1\le i_1<i_2<\ldots<i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$ for any positive real numbers $x_1,x_2,\ldots,x_n$, where $k = 1,2,\ldots,n.$

Since $\frac{1}{\sqrt{k}} < 2\left(\sqrt{k}-\sqrt{k-1}\right)$ then $S_n < \sum_{k=1}^{n}2\left(\sqrt{k}-\sqrt{k-1}\right) = 2\sqrt{n}$.

Also, noting that $S_n - 2\sqrt{n+1}$ increase and $S_n - 2\sqrt{n}$ decrease by $n \in \mathbb{N}$ we can define some constant $c := \lim_{n\to\infty}(S_n - 2\sqrt{n}) = \lim_{n\to\infty}\left(S_n - 2\sqrt{n+1}\right)$.

Thus, denoting $\alpha_n := S_n - 2\sqrt{n} - c$ we obtain $S_n = 2\sqrt{n} + c + \alpha_n$, where $\lim_{n\to\infty}\alpha_n = 0.$

We have $P_n = 1 + \frac{S_n}{n} + \frac{1}{n^2}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}} + R_n$, where $R_n := \sum_{k=3}^{n}\frac{1}{n^k}\sum_{1\le i_1<i_2<\ldots<i_k j\le n}^{n}\frac{1}{\sqrt{i_1\cdot i_2\cdot\ldots\cdot i_k}}$

Applying Maclaurin's inequality $\sqrt[k]{\frac{\sigma_k}{\binom{n}{k}}} \le \frac{\sigma_1}{n}$ in the form $\sigma_k \le \frac{\binom{n}{k}}{n^k}\sigma_1$

to $x_n = \frac{1}{\sqrt{n}}, n \in \mathbb{N}$ we obtain $\frac{1}{\sqrt{i_1\cdot i_2\cdot\ldots\cdot i_k}} \le \frac{\binom{n}{k}}{n^k}\sigma_1^k = \frac{\binom{n}{k}}{n^k}S_n^k$ and, therefore,

$R_n \le \sum_{k=3}^{n}\frac{1}{n^k}\cdot\frac{\binom{n}{k}}{n^k}S_n^k$. Since $\frac{\binom{n}{k}}{n^k} < \frac{1}{k!}$ and $S_n < 2\sqrt{n}$ then

$$R_n < \sum_{k=3}^{n}\frac{1}{n^k k!}\cdot 2^k n^{k/2} \le \frac{1}{n^{3/2}}\sum_{k=3}^{n}\frac{2^k}{k!} < \frac{e^2}{n^{3/2}}.$$

Thus, $0 < P_n - 1 - \frac{S_n}{n} - \frac{1}{n^2}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}} < \frac{e^2}{n^{3/2}} \iff$

$$0 < n(P_n - 1) - S_n - \frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}} < \frac{e^2}{n^{1/2}}$$ and, therefore,

$$\lim_{n\to\infty}\left(n(P_n-1) - S_n - \frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}\right) = 0.$$

Noting that $\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}} = \frac{1}{2}\left(\left(\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\right)^2 - \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}\right)^2\right) = \frac{1}{2}(S_n^2 - H_n)$ and

$$\lim_{n\to\infty}\frac{H_n}{n} = 0, \lim_{n\to\infty}\frac{S_n^2}{n} = \lim_{n\to\infty}\frac{(2\sqrt{n}+c+\alpha_n)^2}{n} = \lim_{n\to\infty}\frac{4n + 4\sqrt{n}(c+\alpha_n) + (c+\alpha_n)^2}{n} = 4$$

we obtain $\lim_{n\to\infty}\frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n}(S_n^2 - H_n) = 2$ and, therefore,

$$\lim_{n\to\infty}\left(n(P_n-1) - S_n - \frac{1}{n}\sum_{1\le i<j\le n}^{n}\frac{1}{\sqrt{ij}}\right) = 0 \iff$$

$$\lim_{n\to\infty}\left(n(P_n - 1) - 2\sqrt{n} - c - \alpha_n - \frac{1}{n} \sum_{1\leq i<j\leq n}^{n} \frac{1}{\sqrt{ij}} \right) = 0 \iff$$

$$\lim_{n\to\infty}(n(P_n - 1) - 2\sqrt{n}\,) = c + 2$$